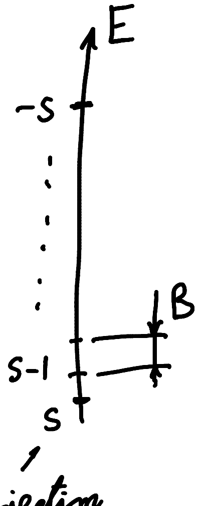
Holstein-Primakoff transformation. Spin waves

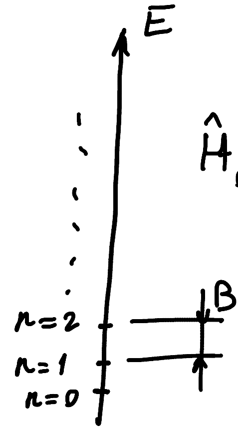
Magnetic excitations (magnons) in solid state behave, in many regards, like particles. We have already learnt the Jordan-Wigner transformation, which maps spins onto fermions in 1D.

Consider a higher spin S in a magnetic field B

$\hat{H} = -\vec{B}\hat{\vec{S}}$

The energies of a harmonic oscillator with frequency B





$$\hat{H}_{osc} = B\left(\hat{b}^{+}\hat{b} + \frac{1}{2}\right) + \text{const}$$

The lowest-energy part of the spectra look identical. If, e.g., the temperature is low, so that only the lowest-energy states contribute, then the dynamics of the spin should be identical to that of a harmonic oscillator.

In principle, we may first try to map them exactly

Holstein-Primakoff transformation

(exact)

$\hat{S}_z = S - \hat{a}^{+}\hat{a}$ ← the bosonic occupation number

Then $S_z = S - n$

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The spin ladder operators decrease and increase the spin projection

$$\hat{S}^+ |S_z\rangle = \sqrt{S(S+1) - S_z(S_z+1)}\ |S_z+1\rangle$$

In terms of the occupation number n

$$\hat{S}^+ |n\rangle = \sqrt{S(S+1) - (S-n)(S-n+1)}\ |n-1\rangle =$$

$$= \sqrt{2Sn - n^2 + n}\ |n-1\rangle =$$

$$= \sqrt{2S}\left(1 - \frac{n-1}{2S}\right)^{\frac{1}{2}} \underbrace{\sqrt{n}\ |n-1\rangle}_{\hat{a}|n\rangle}$$

Thus,

$$\hat{S}^+ = \sqrt{2S}\sqrt{1 - \frac{\hat{a}^+\hat{a}}{2S}}\ \hat{a}$$

Holstein-Primakoff transformation:

$$\boxed{\hat{S}^+ = \sqrt{2S}\left(1 - \frac{\hat{a}^+\hat{a}}{2S}\right)^{\frac{1}{2}}\hat{a}, \quad \hat{S}^- = \sqrt{2S}\ \hat{a}^+\left(1 - \frac{\hat{a}^+\hat{a}}{2S}\right)^{\frac{1}{2}}, \quad S_z = S - \hat{a}^+\hat{a}}$$

This is exact, but allows for an expansion in $1/S$

$$\hat{S} = \sqrt{2S}\left(1 - \frac{\hat{a}^+\hat{a}}{4S} + \dots\right)\hat{a}$$

Has a good track record, however, even for $S = \frac{1}{2}$

Spin waves

Consider the Heisenberg model

$$\hat{H} = -J \sum \hat{\vec{S}}\ \hat{\vec{S}}$$

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$$\hat{H} = -J\sum_{(ij)} \hat{\vec{s}}_i\, \hat{\vec{s}}_j$$

(ij) – Nearest neighbours

$$= -J\sum_{(ij)} \left(s_i^z s_j^z + s_i^x s_j^x + s_i^y s_j^y \right)$$

Everything has to be expressed through s^z, s^+ and s^-

$$s_i^+ s_j^- + s_i^- s_j^+ = (s_i^x + i s_i^y)(s_j^x - i s_j^y) + (s_i^x - i s_i^y)(s_j^x + i s_j^y) =$$
$$= 2 s_i^x s_j^x + 2 s_i^y s_j^y$$

$$\hat{H} = -J\sum_{(ij)} \left[s_i^z s_j^z + \tfrac{1}{2}(s_i^+ s_j^- + s_i^- s_j^+) \right]$$

Assume $J > 0$. Then all spins want to align along the same direction at low temperatures.

Let z be this direction.

$$s_i^z = S - n_i \equiv S - \hat{a}_i^+ \hat{a}_i$$

n_i – Relatively small

The leading-order contribution to the (free) energy $\propto S^2$. Keep the subleading order, $\propto S$.

For that, $\hat{s}_i \approx \sqrt{2S}\, \hat{a}_i$, $\hat{s}_i^+ \approx \sqrt{2S}\, \hat{a}_i^+$

$$\hat{H} \approx -JS^2 \frac{Nz}{2} - SJ\sum_{(ij)} \left[-\hat{a}_i^+ \hat{a}_i - \hat{a}_j^+ \hat{a}_j + \hat{a}_i^+ \hat{a}_j + \hat{a}_j^+ \hat{a}_i \right]$$

Terms $O(1)$ have been dropped

$$\hat{H} = -JS^2 \frac{Nz}{2} + \underbrace{SJz}_{\parallel} \sum_i \hat{a}_i^+ \hat{a}_i - SJ \underbrace{\sum_{(ij)} (\hat{a}_i^+ a_j + \hat{a}_j^+ \hat{a}_i)}$$

B_{eff} – effective magnetic field acting on spin i

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Hopping term $\equiv -sJ \sum_{\substack{i,j \\ i \neq j}} \hat{a}_i^+ \hat{a}_j$

Introduce plane waves $\hat{a}_k = \frac{1}{\sqrt{N}} \sum_{\vec{r}} \hat{a}_{\vec{r}}\, e^{i\vec{k}\vec{r}}$

$$\hat{H} = -Js^2 + \sum_k \omega_k \hat{a}_k^+ \hat{a}_k$$

In 3D (when $z = 6$)

$$\boxed{\omega_k = 2sJ\left(3 - \cos(k_x l) - \cos(k_y l) - \cos(k_z l)\right)}$$

– the dispersion of (ferro-) magnons

For $|kl| \ll 1$

$$\omega_k \approx sJ l^2 k^2$$

<u>Soft modes</u>

The propagation of magnetic waves is similar to the propagation of particles. Magnons are bosons. May be probed, e.g., by X-ray scattering

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Often one uses <u>Dyson-Maleev</u> representation

$$\hat{S}_i^+ = \sqrt{2s}\left(\hat{a}_i - \frac{1}{2s}\hat{a}_i^+ \hat{a}_i \hat{a}_i\right)$$

$$\hat{S}_i^- = \sqrt{2s}\ \hat{a}_i^+$$

$$S_i^z = s - \hat{a}_i^+ \hat{a}_i$$

It is non-Hermitean ($(\hat{S}_i^-)^+ \neq \hat{S}_i^+$), but the commutation relations are met.